

# Some properties of generalized and approximately dual frames in Hilbert spaces

H. JAVANSHIRI

## Abstract

In the present paper, some sufficient and necessary conditions for two frames  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  under which they are approximately or generalized dual frames are determined depending on the properties of their analysis and synthesis operators. We also give a new characterization for approximately dual frames associated with a given frame and given operator by using of bounded operators. Among other things, we prove that if two frames  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  are close to each other, then we can find approximately dual frames  $\Phi^{ad} = (\varphi_n^{ad})_n$  and  $\Psi^{ad} = (\psi_n^{ad})_n$  of them which are close to each other and  $T_\Phi U_{\Phi^{ad}} = T_\Psi U_{\Psi^{ad}}$ , where  $T_\Phi$  and  $T_\Psi$  (resp.  $U_{\Phi^{ad}}$  and  $U_{\Psi^{ad}}$ ) are the analysis operators (resp. synthesis operators) of the frames  $\Phi$  and  $\Psi$  (resp.  $\Phi^{ad}$  and  $\Psi^{ad}$ ), respectively. We then give some consequences on generalized dual frames. Finally, we apply these results to find some construction results for approximately dual frames for a given Gabor frame.

**Mathematics Subject Classification:** Primary: 42C15; Secondary: 47A58.

**Key words:** Frame, Gabor frame, dual frame, approximately dual frame, generalized dual frame.

## 1 Introduction and some prerequisites

Throughout this paper, we denote by  $\mathcal{H}$  a separable Hilbert space with the inner product “ $\langle \cdot, \cdot \rangle$ ” and for basic notations and terminologies on the theory of frames for  $\mathcal{H}$ , we shall follow [5]. Recall that a sequence  $\Phi = (\varphi_n)_n \subseteq \mathcal{H}$  is a *frame* for  $\mathcal{H}$ , if there exist constants  $\mathfrak{m}_\Phi, \mathfrak{M}_\Phi > 0$  such that for all  $f \in \mathcal{H}$

$$\mathfrak{m}_\Phi \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \leq \mathfrak{M}_\Phi \|f\|^2, \quad (1)$$

where  $\mathfrak{m}_\Phi, \mathfrak{M}_\Phi$  are called frame bounds. Moreover, the sequence  $\Phi = (\varphi_n)_n$  is called a *Bessel sequence* for  $\mathcal{H}$ , if only the second inequality of (1) holds. Let also, the space  $\ell^2$  is defined as usual and we denote its canonical orthonormal basis by  $\Delta = (\delta_n)_n$ . Moreover, the notation  $B(\mathcal{H})$  (resp.  $B(\mathcal{H}, \ell^2)$ ) is used to denote the collection of all bounded linear operators from  $\mathcal{H}$  into  $\mathcal{H}$  (resp.  $\ell^2$ ).

It is well-known that the Bessel sequence  $\Phi = (\varphi_n)_n$  gives three operators which plays a crucial role in the theory of frames; In what follows, the notation

•  $U_\Phi : \mathcal{H} \longrightarrow \ell^2$  is used to denote the *analysis operator* of the Bessel sequence  $\Phi$  and defined by  $U_\Phi(f) := (\langle f, \varphi_n \rangle)_n$  for all  $f \in \mathcal{H}$ ,

•  $T_\Phi : \ell^2 \longrightarrow \mathcal{H}$  is used to denote the *synthesis operator* of the Bessel sequence  $\Phi$  and defined by  $T_\Phi((c_n)_n) := \sum_{n=1}^{\infty} c_n \varphi_n$  for all  $f \in \mathcal{H}$ ,

•  $S_\Phi : \mathcal{H} \longrightarrow \mathcal{H}$  is used to denote the *frame operator* of the Bessel sequence  $\Phi$  and defined by  $S_\Phi(f) := \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$  for all  $f \in \mathcal{H}$ .

Note that  $T_\Phi^* = U_\Phi$ , where  $T_\Phi^*$  is the adjoint of the operator  $T_\Phi$ . Furthermore, if  $\Phi$  is a frame, then  $S_\Phi$  is a bounded, invertible, self-adjoint and positive operator such that any  $f \in \mathcal{H}$  can be expressed as

$$f = \sum_{n=1}^{\infty} \langle f, S_\Phi^{-1}(\varphi_n) \rangle \varphi_n = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle S_\Phi^{-1}(\varphi_n); \quad (2)$$

From now on, the notation  $\tilde{\Phi}$  is used to denote the sequence  $(\tilde{\varphi}_n)_n$  which defined by  $\tilde{\varphi}_n := S_\Phi^{-1}(\varphi_n)$  for all  $n \in \mathbb{N}$ , and it is called the *canonical dual frame* of  $\Phi$ . Moreover, for the frame  $\Phi$  which is not a Riesz basis, there exists infinitely many sequences  $\Phi^d = (\varphi_n^d)_n$  such that the following reconstruction formula is hold for all  $f \in \mathcal{H}$

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n^d \rangle \varphi_n = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n^d, \quad (3)$$

see Theorem 5.2.3 of [5]. Recall that the sequence  $\Phi^d$  which satisfies the inequality (3) is called the *dual frame* of  $\Phi$ . In terms of the operators  $T_\Phi$  and  $U_{\Phi^d}$ , the equality (3) means that  $T_\Phi U_{\Phi^d} = Id_{\mathcal{H}} = T_{\Phi^d} U_\Phi$ , where here and in the sequel,  $Id_{\mathcal{H}}$  is the identity operator on  $\mathcal{H}$ .

Approximately dual frames as an applicable and interesting duality principle in the theory of frames was introduced by Christensen and Laugesen in [7]. Here it should be noted that, the idea of approximately dual frames can be found in Gabor systems [2, 13, 24], wavelets [3, 15, 17], coorbit theory [11] and sensor modeling [22], before they introduce it. Moreover, their paper initiated a series of subsequent publications (see for example [3, 8, 10, 12, 14, 20, 21]) and has had a great impact. In fact, they said that two Bessel sequences  $\Phi = (\varphi_n)_n$  and  $\Phi^{ad} = (\varphi_n^{ad})_n$  are approximately dual frames if  $\|Id_{\mathcal{H}} - T_\Phi U_{\Phi^{ad}}\| < 1$  or  $\|Id_{\mathcal{H}} - T_{\Phi^{ad}} U_\Phi\| < 1$ . It follows that, the operator  $\mathcal{A} = T_\Phi U_{\Phi^{ad}}$  is invertible and any  $f \in \mathcal{H}$  can be expressed as

$$f = \sum_{n=1}^{\infty} \langle \mathcal{A}^{-1} f, \varphi_n^{ad} \rangle \varphi_n. \quad (4)$$

Recently, another generalization of duality principle in the theory of frames has been proposed by Dehghan and Hasankhani-Fard [8]. Let us recall from [8] that, the frame  $\Phi^{gd} = (\varphi_n^{gd})_n$  is a generalized dual frame or g-dual frame of  $\Phi$  with corresponding invertible operator (or with invertible operator)  $\mathcal{A} \in B(\mathcal{H})$ , if we have the following inequality for all  $f \in \mathcal{H}$

$$f = \sum_{n=1}^{\infty} \langle \mathcal{A} f, \varphi_n^{gd} \rangle \varphi_n. \quad (5)$$

In terms of the operators  $T_\Phi$  and  $U_{\Phi^{gd}}$ , the equality (5) means that  $T_\Phi U_{\Phi^{gd}} = \mathcal{A}^{-1}$ . It follows that the operator  $\mathcal{A}$  in (5) is unique. Moreover, a simple observation shows that if two frames are approximately dual frames, then they are g-dual frames. Although this is not unexpected, Example 4.1 of [8] illustrated that the set of approximately duals of a frame is a proper subset

of the set of its g-duals. Moreover, recall from [8, Theorem 3.1] that the set of all g-dual frames of the frame  $\Phi$  with corresponding invertible operator  $\mathcal{A}^*S_\Phi^{-1}$  are precisely the sequences of the form

$$\Phi^{gd} = \left( \mathcal{A}\varphi_n + \psi_n - \sum_{m=1}^{\infty} \langle S_\Phi^{-1}\varphi_n, \varphi_m \rangle \psi_m \right)_n,$$

where  $\Psi = (\psi_n)_n$  is a Bessel sequence in  $\mathcal{H}$  and  $\mathcal{A} \in B(\mathcal{H})$  is an invertible operator; In particular  $T_\Phi U_\Psi = \mathcal{A}^*S_\Phi^{-1}$ .

In the present work, we introduce a general method of constructing approximately duals (resp. g-duals) frame for a given frame and given operator by using of bounded operators which our explicit construction can be easily applied for Gabor frames. We then show that for a perturbed frame, one can construct always an approximately dual (resp. g-dual) frame which is close to the approximately dual (resp. g-dual) frame of the original frame. Finally, we show that by choosing an appropriate dual generator for a Gabor frame provides more precise results.

## 2 The basic results

We commence this section with the following result which gives a sufficient and necessary condition for two frames  $\Phi$  and  $\Psi$  under which they are g-dual frames. To this end, recall that every bounded and positive operator  $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$  has a unique bounded and positive square root  $\mathcal{C}^{\frac{1}{2}}$ . Moreover, if the operator  $\mathcal{C}$  is self-adjoint (resp. invertible), then  $\mathcal{C}^{\frac{1}{2}}$  is also self-adjoint (resp. invertible), see for example Lemma 2.4.4 of [5].

**Lemma 2.1** *Let  $\Phi$  and  $\Psi$  be two frames for  $\mathcal{H}$  with Bessel bounds  $M_\Phi$  and  $M_\Psi$ , respectively. Then  $\Psi$  is a g-dual frame of  $\Phi$  if and only if there exists an invertible operator  $\mathcal{D} \in B(\mathcal{H})$  such that  $T_\Phi U_\Psi = S_\Phi^{\frac{1}{2}}\mathcal{D}$  and  $\mathcal{D}\mathcal{D}^* \leq M_\Psi Id_{\mathcal{H}}$ .*

*Proof.* The proof of the “only if” part is trivial. To prove the “if” part, suppose that  $\Psi$  is a g-dual frame of  $\Phi$ . Observe that

$$\frac{1}{M_\Psi} \|T_\Psi U_\Phi f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2, \quad (6)$$

for all  $f \in \mathcal{H}$ ; Indeed,

$$\begin{aligned} \|T_\Psi U_\Phi f\| &= \sup_{\|g\|=1} |\langle T_\Psi U_\Phi f, g \rangle| \\ &= \sup_{\|g\|=1} |\langle f, T_\Phi U_\Psi g \rangle| \\ &= \sup_{\|g\|=1} \left| \langle f, \sum_{n=1}^{\infty} \langle g, \psi_n \rangle \varphi_n \rangle \right| \\ &= \sup_{\|g\|=1} \left| \sum_{n=1}^{\infty} \overline{\langle g, \psi_n \rangle} \langle f, \varphi_n \rangle \right| \\ &\leq \sup_{\|g\|=1} \left( \sum_{n=1}^{\infty} |\langle g, \psi_n \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\|g\|=1} \sqrt{M_\Psi} \|g\| \left( \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{M_\Psi} \left( \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Note that, in terms of the inner product of  $\mathcal{H}$ , the equality (6) means that

$$\left\langle T_\Psi U_\Phi f, T_\Psi U_\Phi f \right\rangle \leq M_\Psi \langle S_\Phi f, f \rangle, \quad (f \in \mathcal{H}).$$

Therefore, we have the following inequality

$$\left\langle (T_\Phi U_\Psi)(T_\Phi U_\Psi)^* f, f \right\rangle \leq M_\Psi \left\langle S_\Phi^{\frac{1}{2}} S_\Phi^{\frac{1}{2}} f, f \right\rangle, \quad (7)$$

for all  $f \in \mathcal{H}$ . We now invoke Theorem 1 of [9] and the invertibility of the operator  $T_\Phi U_\Psi$ , to conclude that there exists an invertible operator  $\mathcal{D} \in B(\mathcal{H})$  such that  $T_\Phi U_\Psi = S_\Phi^{\frac{1}{2}} \mathcal{D}$ . Moreover, inequality (7) means that

$$S_\Phi^{\frac{1}{2}} \mathcal{D} \mathcal{D}^* S_\Phi^{\frac{1}{2}} = (S_\Phi^{\frac{1}{2}} \mathcal{D})(S_\Phi^{\frac{1}{2}} \mathcal{D})^* \leq M_\Psi S_\Phi^{\frac{1}{2}} S_\Phi^{\frac{1}{2}}.$$

Therefore, [23, Theorem 2.2.5(2)] implies that  $\mathcal{D} \mathcal{D}^* \leq M_\Psi Id_{\mathcal{H}}$ . ■

By the same argument as above, we can prove the following result.

**Lemma 2.2** *Let  $\Phi$  and  $\Psi$  be two frames for  $\mathcal{H}$  with Bessel bounds  $M_\Phi$  and  $M_\Psi$ , respectively. Then  $\Psi$  is an approximately dual frame of  $\Phi$  if and only if there exists an operator  $\mathcal{D} \in B(\mathcal{H})$  such that  $T_\Phi U_\Psi = S_\Phi^{\frac{1}{2}} \mathcal{D}$ ,  $\mathcal{D} \mathcal{D}^* \leq M_\Psi Id_{\mathcal{H}}$  and  $\|Id_{\mathcal{H}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| < 1$ .*

From now on, for the frame  $\Phi = (\varphi_n)_n$ , we assume that  $\mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  stands for the set of all right annihilator of the operator  $T_\Phi$  in  $B(\mathcal{H}, \ell^2)$ ; That is,

$$\mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi) := \left\{ \Theta \in B(\mathcal{H}, \ell^2) : T_\Phi \Theta = 0 \right\}.$$

By the use of [5, Theorem 5.2.2] and a routine computation, we can see that the frame  $\Phi$  is a Riesz basis for  $\mathcal{H}$  if and only if  $\mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi) = \{0\}$ .

The following theorem gives a new characterization of all approximately dual frames of  $\Phi = (\varphi_n)_n$  in terms of the operators in  $\mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$ ; In fact, this is a construction result for approximately dual frames associated with a given frame. For the rest of the paper the letter “ $\langle \cdot, \cdot \rangle_{\ell^2}$ ” means the inner product of  $\ell^2$ .

**Theorem 2.3** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$ . Then, the approximately dual frames of  $\Phi$  are precisely the families  $\Phi^{ad} = (\varphi_n^{ad})_n$  such that*

$$\varphi_n^{ad} = \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n + \Theta^*(\delta_n), \quad (n \in \mathbb{N})$$

where  $\Theta \in \mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  and  $\mathcal{D}$  is a bounded operator on  $\mathcal{H}$  for which  $\|Id_{\mathcal{H}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| < 1$ . In particular,  $T_\Phi U_{\Phi^{ad}} = S_\Phi^{\frac{1}{2}} \mathcal{D}$ .

Proof. First, assume that  $\Phi^{ad} = (\varphi_n^{ad})_n$  is a sequence in  $\mathcal{H}$  such that

$$\varphi_n^{ad} = \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n + \Theta^*(\delta_n), \quad (n \in \mathbb{N})$$

where  $\Theta \in \mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  and  $\mathcal{D}$  is a bounded operator on  $\mathcal{H}$  for which  $\|Id_{\mathcal{H}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| < 1$ . Now, since the sequence  $\Phi^{-\frac{1}{2}} := (S_\Phi^{-\frac{1}{2}} \varphi_n)_n$  is a tight frame with frame bound equal to 1, we can see easily that  $\Phi^{ad}$  is a Bessel sequence with Bessel bound  $\mathbf{M}_{\Phi^{ad}} = \|\mathcal{D}\|^2 + \|\Theta\|^2 + 2\|\mathcal{D}\|\|\Theta\|$ . We also observed that

$$T_\Phi U_{\Phi^{ad}} f = S_\Phi^{\frac{1}{2}} \mathcal{D} f;$$

This is because of,

$$\begin{aligned} U_{\Phi^{ad}}(f) &= \left( \langle f, \varphi_n^{ad} \rangle \right)_n = \left( \langle f, \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n \rangle + \langle f, \Theta^*(\delta_n) \rangle \right)_n \\ &= \left( \langle S_\Phi^{-\frac{1}{2}} \mathcal{D} f, \varphi_n \rangle + \langle \Theta f, \delta_n \rangle_{\ell^2} \right)_n \\ &= \left( \langle S_\Phi^{-\frac{1}{2}} \mathcal{D} f, \varphi_n \rangle + (\Theta f)_n \right)_n \\ &= U_\Phi S_\Phi^{-\frac{1}{2}} \mathcal{D} f + \Theta f, \end{aligned}$$

for all  $f \in \mathcal{H}$ , where  $(\Theta f)_n$  is the  $n$ th term of the sequence  $\Theta f$ . Moreover,

$$\langle \mathcal{D} \mathcal{D}^* f, f \rangle = \|\mathcal{D}^* f\|^2 \leq \|\mathcal{D}\|^2 \langle f, f \rangle \leq \mathbf{M}_{\Phi^{ad}} \langle f, f \rangle,$$

for all  $f \in \mathcal{H}$ . We now invoke Lemma 2.2 to conclude that  $\Phi^{ad}$  is an approximately dual frame of  $\Phi$ .

Conversely, let  $\Phi^{ad} = (\varphi_n^{ad})_n$  be an approximately dual frame of  $\Phi = (\varphi_n)_n$  with Bessel bound  $\mathbf{M}_{\Phi^{ad}}$ . Then, in view of Lemma 2.2, there exists an invertible operator  $\mathcal{D}$  in  $B(\mathcal{H})$  such that  $T_\Phi U_{\Phi^{ad}} = S_\Phi^{\frac{1}{2}} \mathcal{D}$ ,  $\mathcal{D} \mathcal{D}^* \leq \mathbf{M}_{\Phi^{ad}} Id_{\mathcal{H}}$  and

$$\|Id_{\mathcal{H}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| = \|Id_{\mathcal{H}} - T_\Phi U_{\Phi^{ad}}\| < 1.$$

Now, define the operator  $\Theta$  from  $\mathcal{H}$  into  $\ell^2$  by

$$\Theta := U_{\Phi^{ad}} - U_\Phi S_\Phi^{-\frac{1}{2}} \mathcal{D}.$$

Observe that the operator  $\Theta$  is in  $\mathbf{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$ . Moreover, for each  $n \in \mathbb{N}$  we have

$$\begin{aligned} \Theta^*(\delta_n) &= T_{\Phi^{ad}}(\delta_n) - \mathcal{D}^* S_\Phi^{-\frac{1}{2}} T_\Phi(\delta_n) \\ &= \varphi_n^{ad} - \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n, \end{aligned}$$

and this completes the proof. ■

As applications of Theorem 2.3, we have the following two results which gives a construction result for approximately dual frames associated with a given frame and given operator.

**Theorem 2.4** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$  with Bessel bound  $M_\Phi$ , and let  $\mathcal{D}$  be an operator in  $B(\mathcal{H})$  such that  $\|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\| < \frac{1}{\sqrt{M_\Phi}}$ . Then, the sequence*

$$\Phi^{ad} = \left( \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n + \Theta^*(\delta_n) \right)_n$$

*is an approximately dual frame of  $\Phi$  for which  $T_\Phi U_{\Phi^{ad}} = S_\Phi^{\frac{1}{2}} \mathcal{D}$ , where  $\Theta$  is an arbitrary element of  $\text{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$ .*

Proof. Note that

$$\begin{aligned} \|Id_{\mathcal{H}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| &= \|S_\Phi^{\frac{1}{2}} S_\Phi^{-\frac{1}{2}} - S_\Phi^{\frac{1}{2}} \mathcal{D}\| \\ &\leq \|S_\Phi^{\frac{1}{2}}\| \|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\| \\ &= \sqrt{\|(S_\Phi^{\frac{1}{2}})^* S_\Phi^{\frac{1}{2}}\|} \|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\| \\ &= \sqrt{\|S_\Phi\|} \|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\| \\ &= \sqrt{M_\Phi} \|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\|. \end{aligned}$$

So, the proof will follow from Theorem 2.3. ■

**Theorem 2.5** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$ , and let  $\mathcal{A}$  be an operator in  $B(\mathcal{H})$  such that  $\|Id_{\mathcal{H}} - \mathcal{A}\| < 1$ . Then, the Bessel sequence  $\Phi^{ad} = (\varphi_n^{ad})_n$  is an approximately dual frame of  $\Phi$  for which  $T_\Phi U_{\Phi^{ad}} = \mathcal{A}$  if and only if there exists  $\Theta \in \text{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  such that  $\varphi_n^{ad} = \mathcal{A}^* \tilde{\varphi}_n + \Theta^*(\delta_n)$  for all  $n \in \mathbb{N}$ .*

The following two corollaries illustrate the construction given in Theorems 2.4 and 2.5, which are very applicable for the construction of approximately dual Gabor frames with a desired approximation rate. Here it should be noted that, if  $\Phi$  and  $\Psi$  are approximately dual frames, then we say that they have approximation rate  $0 < \varepsilon < 1$ , if  $\|Id_{\mathcal{H}} - T_\Phi U_\Psi\| \leq \varepsilon$ .

**Corollary 2.6** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$  with Bessel bound  $M_\Phi$  and a dual frame  $\Phi^d = (\varphi_n^d)_n$ . Then, for each operator  $\mathcal{D}$  in  $B(\mathcal{H})$  such that  $\|S_\Phi^{-\frac{1}{2}} - \mathcal{D}\| < \frac{1}{\sqrt{M_\Phi}}$ , the sequence*

$$\Phi^{ad} = \left( \mathcal{D}^* S_\Phi^{-\frac{1}{2}} \varphi_n - \varphi_n + S_\Phi(\varphi_n^d) \right)_n$$

*is an approximately dual frame of  $\Phi$  for which  $T_\Phi U_{\Phi^{ad}} = S_\Phi^{\frac{1}{2}} \mathcal{D}$ .*

**Corollary 2.7** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$  with a dual frame  $\Phi^d = (\varphi_n^d)_n$ . Let also  $\mathcal{A}$  be a bounded operator on  $\mathcal{H}$  such that  $\|Id_{\mathcal{H}} - \mathcal{A}\| < 1$ . Then, the sequence  $\Phi^{ad} = (\varphi_n^{ad})_n$  defined by*

$$\varphi_n^{ad} = \mathcal{A}^* \tilde{\varphi}_n - \varphi_n + S_\Phi(\varphi_n^d), \quad (n \in \mathbb{N})$$

*is an approximately dual frame of  $\Phi$  for which  $T_\Phi U_{\Phi^{ad}} = \mathcal{A}$ .*

In the following, we use the perturbation idea to construct approximately dual frames. The notation  $\Phi - \Psi$  (resp.  $\Phi^{ad} - \Psi^{ad}$ ) in this theorem denotes the sequence  $(\varphi_n - \psi_n)_n$  (resp.  $(\varphi_n^{ad} - \psi_n^{ad})_n$ ).

**Theorem 2.8** *Let  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  be two frames for  $\mathcal{H}$  and let  $\Phi^{ad} = (\varphi_n^{ad})_n$  be a fixed approximately dual frame of  $\Phi$ . If  $\Phi - \Psi$  is a Bessel sequence with sufficiently small bound  $M_{\Phi-\Psi}$ , then there exists an approximately dual frame  $\Psi^{ad} = (\psi_n^{ad})_n$  of  $\Psi$  such that  $\Phi^{ad} - \Psi^{ad}$  is also a Bessel sequence and its bound is a multiple of  $M_{\Phi-\Psi}$ . In particular,  $T_\Phi U_{\Phi^{ad}} = T_\Psi U_{\Psi^{ad}}$ .*

Proof. In view of Theorems 2.3 and 2.5, there exists an operator  $\mathcal{A}$  in  $B(\mathcal{H})$  and  $\Theta \in \text{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  such that

- (a)  $\|Id_{\mathcal{H}} - \mathcal{A}\| < 1$ ,
- (b)  $\varphi_n^{ad} = \mathcal{A}^* \tilde{\varphi}_n + \Theta^*(\delta_n)$ , for all  $n \in \mathbb{N}$ ,
- (c)  $T_\Phi U_{\Phi^{ad}} = \mathcal{A}$ .

Now, let  $\Omega := (\omega_n)_n$  be a sequence in  $\mathcal{H}$  such that for each  $n \in \mathbb{N}$

$$\omega_n = \mathcal{A}^* \tilde{\psi}_n + \Theta^*(\delta_n).$$

A routine computation shows that the sequence  $\Omega$  is a Bessel sequence. Moreover, for each  $f \in \mathcal{H}$  we have

$$\begin{aligned} \mathcal{A}^{-1} T_\Psi U_\Omega f &= \mathcal{A}^{-1} \left( \sum_{n=1}^{\infty} \langle f, \omega_n \rangle \psi_n \right) \\ &= \mathcal{A}^{-1} \left( \sum_{n=1}^{\infty} \langle f, \mathcal{A}^* \tilde{\psi}_n \rangle \psi_n + \sum_{n=1}^{\infty} \langle f, \Theta^*(\delta_n) \rangle \psi_n \right) \\ &= \mathcal{A}^{-1} \left( \sum_{n=1}^{\infty} \langle \mathcal{A} f, \tilde{\psi}_n \rangle \psi_n + \sum_{n=1}^{\infty} \langle \Theta f, \delta_n \rangle_{\ell^2} \psi_n \right) \\ &= \mathcal{A}^{-1} (\mathcal{A} f + T_\Psi \Theta f) \\ &= f + \mathcal{A}^{-1} T_\Psi \Theta f. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - \mathcal{A}^{-1} T_\Psi U_\Omega f\| &= \|\mathcal{A}^{-1} T_\Psi \Theta f\| = \|\mathcal{A}^{-1} T_\Psi \Theta f - \mathcal{A}^{-1} T_\Phi \Theta f\| \\ &\leq \sqrt{M_{\Phi-\Psi}} \|\Theta\| \|\mathcal{A}^{-1}\| \|f\|, \end{aligned}$$

for all  $f \in \mathcal{H}$ . Hence, the operator  $\mathcal{C} := T_\Omega U_\Psi \mathcal{A}^{*-1}$  is invertible for sufficiently small  $M_{\Phi-\Psi} > 0$ . In particular, the operator  $T_\Psi U_\Omega$  is invertible and for each  $f \in \mathcal{H}$  we have

$$\begin{aligned} f = (T_\Psi U_\Omega)(T_\Psi U_\Omega)^{-1} f &= \sum_{n=1}^{\infty} \langle (T_\Psi U_\Omega)^{-1} f, \omega_n \rangle \psi_n \\ &= \sum_{n=1}^{\infty} \langle f, (T_\Omega U_\Psi)^{-1} \omega_n \rangle \psi_n \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \langle \mathcal{A}^{-1}f, (T_{\Omega}U_{\Psi}\mathcal{A}^{*-1})^{-1}\omega_n \rangle \psi_n \\
&= \sum_{n=1}^{\infty} \langle \mathcal{A}^{-1}f, \mathcal{C}^{-1}\omega_n \rangle \psi_n.
\end{aligned}$$

It follows that  $\Psi^{ad} := (\mathcal{C}^{-1}\omega_n)_n$  is a frame for  $\mathcal{H}$  such that  $T_{\Psi}U_{\Psi^{ad}} = \mathcal{A}$  and therefore  $\Psi^{ad}$  is an approximately dual frame of  $\Psi$ .

Now, the proof will be completed if we show that  $\Phi^{ad} - \Psi^{ad}$  is also a Bessel sequence and its bound is a multiple of  $M_{\Phi-\Psi}$ . To this end, we note that

$$\|\mathcal{C}^{-1}\| \leq \frac{1}{1 - \|Id_{\mathcal{H}} - \mathcal{C}\|} \leq \frac{1}{1 - \sqrt{M_{\Phi-\Psi}} \|\Theta\| \|\mathcal{A}^{-1}\|}, \quad (8)$$

$$\|Id_{\mathcal{H}} - \mathcal{C}^{-1}\| \leq \|\mathcal{C}^{-1}\| \|Id_{\mathcal{H}} - \mathcal{C}\| \leq \frac{\sqrt{M_{\Phi-\Psi}} \|\Theta\| \|\mathcal{A}^{-1}\|}{1 - \sqrt{M_{\Phi-\Psi}} \|\Theta\| \|\mathcal{A}^{-1}\|}, \quad (9)$$

and

$$\begin{aligned}
\|S_{\Phi} - S_{\Psi}\| &= \|T_{\Phi}U_{\Phi} - T_{\Phi}U_{\Psi} + T_{\Phi}U_{\Psi} - T_{\Psi}U_{\Psi}\| \\
&\leq \|T_{\Phi} - T_{\Psi}\| (\|T_{\Phi}\| + \|T_{\Psi}\|) \\
&\leq \sqrt{M_{\Phi-\Psi}} (\sqrt{M_{\Phi}} + \sqrt{M_{\Psi}})
\end{aligned}$$

Moreover, for all  $(c_n)_n \in \ell^2$  and all  $f \in \mathcal{H}$  we have

$$\begin{aligned}
\langle T_{\Phi^{ad}}((c_n)_n), f \rangle - \langle T_{\Omega}((c_n)_n), f \rangle &= \langle (c_n)_n, U_{\Phi^{ad}}f \rangle_{\ell^2} - \langle (c_n)_n, U_{\Omega}f \rangle_{\ell^2} \\
&= \langle (c_n)_n, U_{\Phi}S_{\Phi}^{-1}\mathcal{A}f \rangle_{\ell^2} - \langle (c_n)_n, U_{\Psi}S_{\Psi}^{-1}\mathcal{A}f \rangle_{\ell^2} \\
&= \langle T_{\Phi}((c_n)_n), S_{\Phi}^{-1}\mathcal{A}f \rangle - \langle T_{\Psi}((c_n)_n), S_{\Psi}^{-1}\mathcal{A}f \rangle \\
&= \langle T_{\Phi}((c_n)_n), (S_{\Phi}^{-1} - S_{\Psi}^{-1})\mathcal{A}f \rangle \\
&\quad + \langle (T_{\Phi} - T_{\Psi})((c_n)_n), S_{\Psi}^{-1}\mathcal{A}f \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T_{\Phi^{ad}}((c_n)_n) - T_{\Omega}((c_n)_n)\| &= \sup_{\|g\|=1} \left| \langle T_{\Phi^{ad}}((c_n)_n) - T_{\Omega}((c_n)_n), g \rangle \right| \\
&= \sup_{\|g\|=1} \left| \langle T_{\Phi}((c_n)_n), (S_{\Phi}^{-1} - S_{\Psi}^{-1})\mathcal{A}g \rangle \right. \\
&\quad \left. + \langle T_{\Phi}((c_n)_n) - T_{\Psi}((c_n)_n), S_{\Psi}^{-1}\mathcal{A}g \rangle \right| \\
&\leq \|S_{\Phi}^{-1} - S_{\Psi}^{-1}\| \|\mathcal{A}\| \|T_{\Phi}((c_n)_n)\| \\
&\quad + \|S_{\Psi}^{-1}\| \|\mathcal{A}\| \|T_{\Phi}((c_n)_n) - T_{\Psi}((c_n)_n)\| \\
&\leq \|S_{\Psi}^{-1}\| \|S_{\Phi} - S_{\Psi}\| \|S_{\Phi}^{-1}\| \|\mathcal{A}\| \|T_{\Phi}((c_n)_n)\| \\
&\quad + \|S_{\Psi}^{-1}\| \|\mathcal{A}\| \|T_{\Phi}((c_n)_n) - T_{\Psi}((c_n)_n)\| \\
&\leq \sqrt{M_{\Phi-\Psi}} \left( \frac{1}{m_{\Phi}m_{\Psi}} (M_{\Phi} + \sqrt{M_{\Psi}M_{\Phi}}) \|\mathcal{A}\| + \frac{1}{m_{\Psi}} \|\mathcal{A}\| \right),
\end{aligned}$$



where,  $\mathfrak{m}_\Phi, \mathbf{M}_\Phi$  and  $\mathfrak{m}_\Psi, \mathbf{M}_\Psi$  are the bounds of the frames  $\Phi$  and  $\Psi$ , respectively. It follows that the sequence  $\Phi^{ad} - \Psi^{ad}$  is a Bessel sequence with Bessel bound

$$\mathbf{M}_{\Phi^{ad} - \Psi^{ad}} := \mathbf{M}_{\Phi - \Psi} \left( \frac{\|\mathcal{A}^{-1}\|}{1 - \sqrt{\mathbf{M}_{\Phi - \Psi}} \|\Theta\| \|\mathcal{A}^{-1}\|} \right)^2 \left( \|\Theta\| \sqrt{\mathbf{M}_{\Phi^{ad}}} + \frac{\|\mathcal{A}\| (m_\Phi + M_\Phi + \sqrt{M_\Psi M_\Phi})}{m_\Phi m_\Psi} \right)^2$$

and this completes the proof; Indeed,

$$\begin{aligned} \|T_{\Phi^{ad}}((c_n)_n) - T_{\Psi^{ad}}((c_n)_n)\| &= \|T_{\Phi^{ad}}((c_n)_n) - T_{\Psi^{ad}}((c_n)_n) - \mathcal{C}^{-1}T_{\Phi^{ad}}((c_n)_n) + \mathcal{C}^{-1}T_{\Phi^{ad}}((c_n)_n)\| \\ &= \|T_{\Phi^{ad}}((c_n)_n) - \mathcal{C}^{-1}T_{\Omega}((c_n)_n) - \mathcal{C}^{-1}T_{\Phi^{ad}}((c_n)_n) + \mathcal{C}^{-1}T_{\Phi^{ad}}((c_n)_n)\| \\ &\leq \|\text{Id}_{\mathcal{H}} - \mathcal{C}^{-1}\| \|T_{\Phi^{ad}}((c_n)_n)\| + \|\mathcal{C}^{-1}\| \|T_{\Phi^{ad}}((c_n)_n) - T_{\mathcal{F}}((c_n)_n)\| \end{aligned}$$

for all  $(c_n)_n \in \ell^2$ . ■

An argument similar to the proof of Theorems 2.3, 2.5 and 2.8 with the aid of Lemma 2.1 gives the following generalization of that theorems. The details are omitted.

**Theorem 2.9** *Let  $\Phi = (\varphi_n)_n$  be a frame for  $\mathcal{H}$  and let  $\mathcal{A}$  be an invertible operator in  $B(\mathcal{H})$ . Then  $\Phi^{gd} = (\varphi_n^{gd})_n$  is a  $g$ -dual frame of  $\Phi$  with corresponding invertible operator  $\mathcal{A} \in B(\mathcal{H})$  if and only if there exists  $\Theta \in \text{ran}_{B(\mathcal{H}, \ell^2)}(T_\Phi)$  such that  $\varphi_n^{gd} = \mathcal{A}^{-1*} \tilde{\varphi}_n + \Theta^*(\delta_n)$  for all  $n \in \mathbb{N}$ .*

**Theorem 2.10** *Let  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  be two frames for  $\mathcal{H}$  and let  $\Phi^{gd} = (\varphi_n^{gd})_n$  be a fixed  $g$ -dual frame of  $\Phi$  with corresponding invertible operator  $\mathcal{A}$ . If  $\Phi - \Psi$  is a Bessel sequence with sufficiently small bound  $\mathbf{M}_{\Phi - \Psi}$ , then there exists a  $g$ -dual frame  $\Psi^{gd} = (\psi_n^{gd})_n$  of  $\Psi$  such that  $\Phi^{gd} - \Psi^{gd}$  is also a Bessel sequence and its bound is a multiple of  $\mathbf{M}_{\Phi - \Psi}$ . In particular,  $T_\Phi U_{\Phi^{gd}} = \mathcal{A}^{-1} = T_\Psi U_{\Psi^{gd}}$ .*

Since each frame  $\Phi$  is a  $g$ -dual frame for itself with corresponding invertible operator  $S_\Phi^{-1}$ , hence an argument similar to the proof of Theorem 2.8 with the aid of Theorem 2.10 gives the following result.

**Theorem 2.11** *Let  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  be two frames for  $\mathcal{H}$ . Let also  $\mathbf{M}_{\Phi - \Psi}$  is the Bessel bound of the Bessel sequence  $\Phi - \Psi$ . Then there exists a  $g$ -dual frame  $\Psi^{gd} = (\psi_n^{gd})_n$  of  $\Psi$  such that  $\Phi - \Psi^{gd}$  is also a Bessel sequence, its bound is a multiple of  $\mathbf{M}_{\Phi - \Psi}$  and  $T_\Psi U_{\Psi^{gd}} = S_\Phi$ .*

Now, let  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  are Riesz bases with Bessel bounds  $\mathbf{M}_\Phi$  and  $\mathbf{M}_\Psi$ , respectively. Then, there exists an invertible operator  $\mathcal{D}$  on  $\mathcal{H}$  such that  $\mathcal{D}\varphi_n = \psi_n$  for all  $n \in \mathbb{N}$ . Observe that the sequence  $\Phi - \Psi$  is a Bessel sequence with Bessel bound

$$\mathbf{M}_{\Phi - \Psi} = \min \left\{ \mathbf{M}_\Phi \| \text{Id}_{\mathcal{H}} - \mathcal{D} \|^2, \mathbf{M}_\Psi \| \text{Id}_{\mathcal{H}} - \mathcal{D}^{-1} \|^2 \right\};$$

Indeed, if  $(c_n)_n$  is an arbitrary element of  $\ell^2$ , then on the one hand we have

$$\begin{aligned} \|(T_\Phi - T_\Psi)((c_n)_n)\| &= \left\| \sum_{n=1}^{\infty} c_n(\varphi_n - \mathcal{D}(\varphi_n)) \right\| \\ &\leq \| \text{Id}_{\mathcal{H}} - \mathcal{D} \| \|T_\Phi((c_n)_n)\| \\ &\leq \sqrt{\mathbf{M}_\Phi} \| \text{Id}_{\mathcal{H}} - \mathcal{D} \| \| (c_n)_n \|_2, \end{aligned}$$

and on the other hand, by the same argument we can see that

$$\|(T_\Phi - T_\Psi)((c_n)_n)\| \leq \sqrt{\mathbf{M}_\Psi} \|Id_{\mathcal{H}} - \mathcal{D}^{-1}\| \|(c_n)_n\|_2.$$

Hence as an immediate corollary from Theorem 2.11, we have the following result.

**Theorem 2.12** *Let  $\Phi = (\varphi_n)_n$  and  $\Psi = (\psi_n)_n$  be two Riesz bases for  $\mathcal{H}$  with Bessel bounds  $\mathbf{M}_\Phi$  and  $\mathbf{M}_\Psi$ , respectively. Then  $\Phi - \Psi$  is a Bessel sequence with Bessel bound  $\mathbf{M}_{\Phi-\Psi} = \min\{\mathbf{M}_\Phi \|Id_{\mathcal{H}} - \mathcal{D}\|^2, \mathbf{M}_\Psi \|Id_{\mathcal{H}} - \mathcal{D}^{-1}\|^2\}$ , where  $\mathcal{D}$  is an operator such that  $\mathcal{D}\varphi_n = \psi_n$  for all  $n \in \mathbb{N}$ . Moreover, there exists a  $g$ -dual frame  $\Psi^{gd} = (\psi_n^{gd})_n$  of  $\Psi$  such that  $\Phi - \Psi^{gd}$  is also a Bessel sequence, its bound is a multiple of  $\mathbf{M}_{\Phi-\Psi}$  and  $T_\Psi U_{\Psi^{gd}} = S_\Phi$ . In particular,  $\Psi^{gd}$  is a Riesz basis.*

Following Balan [1], we say that two frames  $\Phi$  and  $\Psi$  for  $\mathcal{H}$  are equivalent frames, if there exists a bounded invertible operator  $Q : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\varphi_n = Q(\psi_n)$  for all  $n \in \mathbb{N}$  or equivalently  $\text{Range}(U_\Phi) = \text{Range}(U_\Psi)$ . Furthermore, the frame  $\Psi$  is partial equivalent with the frame  $\Phi$ , if there exists a bounded operator  $Q : \mathcal{H} \rightarrow \mathcal{H}$  (not necessarily invertible) such that  $\varphi_n = Q(\psi_n)$  for all  $n \in \mathbb{N}$  or equivalently  $\text{Range}(U_\Phi) \subsetneq \text{Range}(U_\Psi)$ . Observe that, in the case where  $\text{Range}(U_\Phi) \subseteq \text{Range}(U_\Psi)$ , then  $T_\Psi^{-1} U_\Phi$  is a right inverse of  $T_\Phi U_\Psi$ ; Indeed, if  $f$  is an arbitrary element of  $\mathcal{H}$ . Then, there exists  $h \in \mathcal{H}$  such that  $U_\Phi S_\Phi^{-1} f = U_\Psi h$ , and therefore

$$\begin{aligned} T_\Phi U_\Psi T_\Psi^{-1} U_\Phi(f) &= T_\Phi U_\Psi S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1}(f) \\ &= T_\Phi U_\Psi S_\Psi^{-1} T_\Psi U_\Psi(h) \\ &= T_\Phi U_\Psi(h) \\ &= T_\Phi U_\Phi S_\Phi^{-1} f = f. \end{aligned}$$

So we have the following result for equivalent frames.

**Proposition 2.13** *Let  $\Phi$  and  $\Psi$  be two frames for  $\mathcal{H}$  such that  $\text{Range}(U_\Phi) = \text{Range}(U_\Psi)$ . Then  $\Phi$  and  $\Psi$  are  $g$ -dual frames. In particular,  $(T_\Phi U_\Psi)^{-1} = T_\Psi^{-1} U_\Phi$ .*

We conclude this section by the following result which is of interest in its own right.

**Proposition 2.14** *Let  $\Phi$  and  $\Psi$  be two frames for  $\mathcal{H}$  such that  $\text{Range}(U_\Phi) \subsetneq \text{Range}(U_\Psi)$  or  $\text{Range}(U_\Psi) \subsetneq \text{Range}(U_\Phi)$ . Then  $\Phi$  and  $\Psi$  are not  $g$ -dual frames. In particular, they are not approximately dual frames.*

*Proof.* Let  $\Phi$  and  $\Psi$  be two frames for  $\mathcal{H}$  such that  $\text{Range}(U_\Phi) \subsetneq \text{Range}(U_\Psi)$ . As we saw in above discussion, this condition implies that  $T_\Psi^{-1} U_\Phi$  is a right inverse of  $T_\Phi U_\Psi$ . Now we show that it is not a left inverse for  $T_\Phi U_\Psi$ . To this end, assume that  $(c_n)_n$  is an element of  $\text{Range}(U_\Psi)$  such that  $(c_n)_n \notin \text{Range}(U_\Phi)$ . Then, there exists  $g_1 \in \mathcal{H}$  and  $0 \neq (d_n)_n \in \ker(T_\Phi)$  such that  $(c_n)_n = U_\Phi g_1 + (d_n)_n$ ; This is because of  $\ell^2 = \text{Range}(U_\Phi) \oplus \ker(T_\Phi)$ . It follows that  $(d_n)_n = U_\Phi g_1 - (c_n)_n \in \text{Range}(U_\Psi) \subseteq \ell^2 \setminus \ker(T_\Psi)$ . This, together with the fact that  $S_\Psi$  is an

invertible operator, implies that  $S_\Psi^{-1}T_\Psi((d_n)_n) \neq 0$ . Now, if  $f'$  is an element of  $\mathcal{H}$  such that  $(c_n)_n = U_\Psi f'$ , then

$$\begin{aligned}
T_\Psi U_\Phi T_\Phi U_\Psi(f') &= S_\Psi^{-1}T_\Psi U_\Phi S_\Phi^{-1}T_\Phi U_\Psi(f') \\
&= S_\Psi^{-1}T_\Psi U_\Phi S_\Phi^{-1}T_\Phi((c_n)_n) \\
&= S_\Psi^{-1}T_\Psi U_\Phi S_\Phi^{-1}T_\Phi(U_\Phi(g_1) + (d_n)_n) \\
&= S_\Psi^{-1}T_\Psi(U_\Phi(g_1)) + S_\Psi^{-1}T_\Psi U_\Phi S_\Phi^{-1}T_\Phi((d_n)_n) \\
&= S_\Psi^{-1}T_\Psi((c_n)_n - (d_n)_n) + S_\Psi^{-1}T_\Psi U_\Phi S_\Phi^{-1}T_\Phi((d_n)_n) \\
&= S_\Psi^{-1}T_\Psi((c_n)_n - (d_n)_n) \\
&= f' - S_\Psi^{-1}T_\Psi((d_n)_n) \neq f',
\end{aligned}$$

and this completes the proof. ■

### 3 Application to Gabor frames

In order to state the results of this section we need to recall the definition and some basic results on the duality conditions for a pair of Gabor systems which have found more and more applications in modern life, signal analysis and many other parts of applied mathematics.

A *Gabor frame* is a frame for  $L^2(\mathbb{R})$  of the form  $\mathcal{G} := (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$ , where  $a, b > 0$  are given,  $g \in L^2(\mathbb{R})$  is a fixed function,  $T_{na}f(x) = f(x - na)$  and  $E_{mb}f(x) = e^{2imbx}f(x)$  for all  $f \in L^2(\mathbb{R})$ . In view of [5, Theorem 9.1.12], the sequence  $\mathcal{G}$  can only be a frame if  $ab \leq 1$ , but it is not a sufficient condition. Moreover, if  $\mathcal{G}$  is a frame and  $ab < 1$ , then there exists infinitely many  $g^d$  in  $L^2(\mathbb{R})$  such that we have the following reconstruction formula for each  $f \in L^2(\mathbb{R})$

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g^d \rangle E_{mb}T_{na}g;$$

That is, the Gabor frames  $\mathcal{G}$  and  $\mathcal{G}^d = (E_{mb}T_{na}g^d)_{m,n \in \mathbb{Z}}$  are dual frames. But the standard choice of  $g^d$  is  $S_{\mathcal{G}}^{-1}g$ , where  $S_{\mathcal{G}} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  defined by

$$S_{\mathcal{G}}f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g, \quad (f \in L^2(\mathbb{R}))$$

is the frame operator of  $\mathcal{G}$ . There are some interesting results for duality conditions of a Gabor frame  $\mathcal{G}$ , provided that the function  $g \in L^2(\mathbb{R})$  has compact support, for more details see [4, 5, 6, 24].

We begin the presentation of the results of this section by recalling the following two results from [4, 6, 18], each of which will be required in our present investigation.

**Proposition 3.1** *Let  $N \in \mathbb{N}$  and  $g \in L^2(\mathbb{R})$  be a function with support in  $[0, N]$ . Let also,  $a, b > 0$  be given,  $b \leq \frac{1}{N}$  and there exists  $m, M > 0$  such that*

$$am \leq \mathbf{G}(x) := \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq bM \quad \text{a.e. } x \in \mathbb{R}.$$

Then  $\mathcal{G} = (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$  is a frame for  $L^2(\mathbb{R})$  with frame bounds  $m, M$ . Moreover, the frame operator  $S_{\mathcal{G}}$  and its inverse  $S_{\mathcal{G}}^{-1}$  are given by

$$S_{\mathcal{G}}(f) = \frac{b}{\mathbf{G}}f \quad \text{and} \quad S_{\mathcal{G}}^{-1}(f) = \frac{\mathbf{G}}{b}f,$$

for all  $f \in L^2(\mathbb{R})$ .

The duality condition for a pair of Gabor systems  $\mathcal{G} = (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}^d = (E_{mb}T_{na}g^d)_{m,n \in \mathbb{Z}}$  is presented by Janssen [18] as follows.

**Lemma 3.2** *Two Bessel sequences  $\mathcal{G} = (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}^d = (E_{mb}T_{na}g^d)_{m,n \in \mathbb{Z}}$  form dual frames for  $L^2(\mathbb{R})$  if and only if*

$$\sum_{k \in \mathbb{Z}} \overline{g(x - n/b - kn)} h(x - ka) = b\delta_{n,0}$$

for almost everywhere  $x$  in  $[0, a]$ .

The proof of the next theorem can be found in the paper by Christensen and Kim [6].

**Theorem 3.3** *Let  $N \in \mathbb{N}$ ,  $b \in (0, \frac{1}{2N-1}]$  and  $g$  be a bounded real-valued function for which*

$$\sum_{n \in \mathbb{Z}} g(x - n) = 1 \quad \text{and} \quad \text{supp } g \subseteq [0, N].$$

Then the function  $g_1^d$  and  $g_2^d$  defined by

$$g_1^d(x) = bg(x) + 2b \sum_{n=1}^{N-1} g(x+n) \quad \text{and} \quad g_2^d(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b \quad \text{and} \quad a_n + a_{-n} = 2b \quad \text{for each } n = 1, 2, \dots, N-1$$

generate two dual frames  $\mathcal{G}_1^d = (E_{mb}T_n g_1^d)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}_2^d = (E_{mb}T_n g_2^d)_{m,n \in \mathbb{Z}}$  for  $\mathcal{G} = (E_{mb}T_n g)_{m,n \in \mathbb{Z}}$ .

Here it should be noted that, Lemma 9.3.1 of [5] guarantees that there are infinitely many operator  $\mathcal{A}$  on  $L^2(\mathbb{R})$  such that it commute with  $E_{\pm b}$ ,  $T_{\pm a}$  and  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$ , where  $a$  and  $b$  are positive real numbers; In fact, each operator of the form

$$S_{\mathcal{L}}f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}l \rangle E_{mb}T_{na}l, \quad (f \in L^2(\mathbb{R}))$$

gives an operator  $\mathcal{A}$  on  $L^2(\mathbb{R})$  such that  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$  and it commute with  $E_{\pm b}$  and  $T_{\pm a}$ , where  $l$  is a function in  $L^2(\mathbb{R})$  such that the sequence  $\mathcal{L} = (E_{mb}T_{na}l)_{m,n \in \mathbb{Z}}$  is a frame for  $L^2(\mathbb{R})$ . Moreover, we should mention that if an operator  $\mathcal{A}$  commute with  $E_{\pm b}$  and  $T_{\pm a}$ , then  $\mathcal{A}$  and its adjoint commute with  $E_{mb}$  and  $T_{na}$  for all  $m, n \in \mathbb{Z}$ . Moreover, we have the following example.

**Example 3.4** Assume that  $\mathcal{G} = (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}^{ad} = (E_{mb}T_{na}g^{ad})_{m,n \in \mathbb{Z}}$  are approximately dual frames. If we set  $\mathcal{A} := T_{\mathcal{G}}U_{\mathcal{G}^{ad}}$ , then  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$  and by a method similar to the proof of Lemma 9.3.1 of [5] one can easily see that  $\mathcal{A}$  and its adjoint commute with  $E_{mb}$  and  $T_{na}$  for all  $m, n \in \mathbb{Z}$ .

Now, in view of Corollary 2.7, we have the following result for the construction of approximately dual frames associated with a given Gabor frame and given operator.

**Proposition 3.5** *Let  $\mathcal{G} = (E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$  be a frame for  $L^2(\mathbb{R})$  and let  $\mathcal{G}^d = (E_{mb}T_{na}g^d)_{m,n \in \mathbb{Z}}$  be an arbitrary dual Gabor frame of  $\mathcal{G}$ . Suppose also that  $\mathcal{A}$  is an operator on  $L^2(\mathbb{R})$  such that  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$  and it commutes with  $E_{\pm b}$  and  $T_{\pm a}$ . Then the family  $\mathcal{G}^{ad} = (E_{mb}T_{na}g^{ad})_{m,n \in \mathbb{Z}}$  is an approximately dual frame of  $\mathcal{G}$  for which  $T_{\mathcal{G}}U_{\mathcal{G}^{ad}} = \mathcal{A}$ , where  $g^{ad} = \mathcal{A}^*S_{\mathcal{G}}^{-1}g - g + S_{\mathcal{G}}(g^d)$ .*

It is well known from [19] that in the case where  $a \leq c \leq 1$ , then  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$  is a Gabor frame, where  $\chi_{[0,c]}$  denotes the characteristic function of the interval  $[0, c]$  on  $\mathbb{R}$ . Moreover, it was shown by Hasankhani-Fard and Dehghan [16, Corollary 2.1.] that two Bessel sequence  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}_{a,c'}^d = (E_mT_{na}\chi_{[0,c']})_{m,n \in \mathbb{Z}}$  are dual frames for  $L^2(\mathbb{R})$  if and only if  $c \leq 1$ ,  $c' \leq 1$  and  $a = \min\{c, c'\}$ . From this, with the aid of Proposition 3.5 above, we can obtain the following explicit construction of approximately dual frames associated with the Gabor frame  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$  for certain choice of  $a$  and  $c$ . In the sequel, for Gabor frame  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$ , the notation  $S_{a,c}$  is used to denote its frame operator and  $M_{a,c}$  denotes its Bessel bound.

**Example 3.6** Let  $a, c, c'$  and  $c''$  be positive numbers such that

$$a \leq c, c' \text{ and } c'' \leq 1, \quad \text{and} \quad a = \min\{c, c'\}.$$

(a) Then two frame  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}_1^{ad} = (E_mT_{na}g_1^{ad})_{m,n \in \mathbb{Z}}$  are approximately dual frames such that  $T_{\mathcal{G}_{a,c}}U_{\mathcal{G}^{ad}} = \frac{1}{M_{a,c''}}S_{a,c''}$ , where

$$g_1^{ad} = \frac{1}{M_{a,c''}} S_{a,c''} S_{a,c}^{-1}(\chi_{[0,c]}) - \chi_{[0,c]} + S_{a,c}(\chi_{[0,c']}).$$

(b) Suppose that  $\mathcal{A}$  is an operator on  $L^2(\mathbb{R})$  such that  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$  and it commutes with  $E_{\pm 1}$  and  $T_{\pm a}$ . Then the sequence  $\mathcal{G}_2^{ad} = (E_mT_{na}g_2^{ad})_{m,n \in \mathbb{Z}}$  is an approximately dual frame of  $\mathcal{G}_{a,c} = (E_mT_{na}\chi_{[0,c]})_{m,n \in \mathbb{Z}}$  for which  $T_{\mathcal{G}_{a,c}}U_{\mathcal{G}^{ad}} = \mathcal{A}$ , where

$$g_2^{ad} = \mathcal{A}^* S_{a,c}^{-1}(\chi_{[0,c]}) - \chi_{[0,c]} + S_{a,c}(\chi_{[0,c']}).$$

(c) Let  $\mathcal{G} = (E_mT_{na}g)_{m,n \in \mathbb{Z}}$  be a frame for  $L^2(\mathbb{R})$  and let  $\mathcal{G}^d = (E_mT_{na}g^d)_{m,n \in \mathbb{Z}}$  be an arbitrary dual Gabor frame of  $\mathcal{G}$ . Suppose also that

$$g_{a,e}^{ad} = \frac{1}{M_{a,e}} S_{a,e} S_{\mathcal{G}}^{-1}g - g + S_{\mathcal{G}}(g^d),$$

then the sequence  $\mathcal{G}_{a,e}^{ad} = (E_mT_{na}g_{a,e}^{ad})_{m,n \in \mathbb{Z}}$  is an approximately dual frame of  $\mathcal{G}$  for which  $T_{\mathcal{G}}U_{\mathcal{G}_{a,e}^{ad}} = \frac{1}{M_{a,e}}S_{a,e}$ , where  $e$  can be any positive real number such that  $a \leq e \leq 1$ .

Recall from [5, Section 6.1] that for each  $N \in \mathbb{N}$  the  $B$ -splines  $B_N$  are given inductively by

$$B_1 = \chi_{[0,1]} \quad \text{and} \quad B_{N+1} = B_N * B_1.$$

The  $B$ -spline  $B_N$  has support on the interval  $[0, N]$ . Furthermore, for each  $N \in \mathbb{N}$ , the sequence  $\mathcal{G}_N = (E_{mb}T_n B_N)_{m,n \in \mathbb{Z}}$  is a Gabor frame with the frame operator

$$S_N f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_n B_N \rangle E_{mb}T_n B_N, \quad (f \in L^2(\mathbb{R})) \quad (10)$$

where  $b \in (0, \frac{1}{2N-1}]$ . Moreover, Theorem 3.3 and [5, Theorem 6.1.1] implies that the functions  $B_{1,N}^d$  and  $B_{2,N}^d$  defined by

$$B_{1,N}^d(x) = bB_N(x) + 2b \sum_{n=1}^{N-1} B_N(x+n) \quad (11)$$

and

$$B_{2,N}^d(x) = \sum_{n=-N+1}^{N-1} a_n B_N(x+n), \quad (12)$$

where

$$a_0 = b \quad \text{and} \quad a_n + a_{-n} = 2b \quad \text{for each } n = 1, 2, \dots, N-1,$$

generate two dual frames  $\mathcal{G}_{1,N}^d = (E_{mb}T_n B_{1,N}^d)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}_{2,N}^d = (E_{mb}T_n B_{2,N}^d)_{m,n \in \mathbb{Z}}$  for  $\mathcal{G}_N = (E_{mb}T_n B_N)_{m,n \in \mathbb{Z}}$ .

As an application of Proposition 3.5 with the aid of Example 6.1 of [7] one can easily obtain the following result for the second order B-spline.

**Example 3.7** Let  $\varphi(x) = e^{-4x^2}$  and

$$g(x) = \frac{15.1}{315} \frac{1}{\sum_{n \in \mathbb{Z}} |B_8(2.36(x+n))|^2} B_8(2.36x).$$

Then, by Example 6.1 of [7], we deduce that the frames  $\Phi = (E_{0.1m}T_n \varphi)_{m,n \in \mathbb{Z}}$  and  $\mathcal{G} = (E_{0.1m}T_n g)_{m,n \in \mathbb{Z}}$  are approximately dual frames and  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| \leq 0.009$ . If now we consider  $\mathcal{A} := T_\Phi U_{\mathcal{G}}$ ,

$$B_2^{ad} = \mathcal{A}^* S_2^{-1} B_2 - B_2 + S_2(B_{1,2}^d)$$

and

$$B_{1,2}^d(x) = 0.1B_2(x) + 0.2B_2(x+1),$$

then the sequence  $\mathcal{G}_2^{ad} = (E_{0.1m}T_n B_2^{ad})_{m,n \in \mathbb{Z}}$  is an approximately dual frame of  $\mathcal{G}_2$  for which  $T_{\mathcal{G}_2} U_{\mathcal{G}_2^{ad}} = \mathcal{A}$ . In particular, the approximation rate is  $\varepsilon = 0.009$ .

Finally, with the above notations on the  $N$ -order B-spline, we have the following practical example.

**Example 3.8** Let  $N \in \mathbb{N}$ ,  $b \in (0, \frac{1}{2N-1}]$  and  $\mathcal{A}$  be an operator on  $L^2(\mathbb{R})$  such that  $\|Id_{L^2(\mathbb{R})} - \mathcal{A}\| < 1$  and it commutes with  $E_{\pm b}$  and  $T_{\pm 1}$ . Let also  $S_N$  be given by equation (10) and  $\mathbf{M}_N$  be the Bessel bound of the frame  $\mathcal{G}_N = (E_{mb}T_n B_N)_{m,n \in \mathbb{Z}}$ .

(a) Then the function  $B_{1,N}^{ad}$  and  $B_{2,N}^{ad}$  defined by

$$B_{1,N}^{ad}(x) = \mathcal{A}^* S_N^{-1} B_N - B_N + S_N(B_{1,N}^d)$$

and

$$B_{2,N}^{ad}(x) = \mathcal{A}^* S_N^{-1} B_N - B_N + S_N(B_{2,N}^d),$$

generate two approximately dual frames  $\mathcal{G}_{1,N}^{ad} = (E_{mb}T_n B_{1,N}^{ad})_{m,n \in \mathbb{Z}}$  and  $\mathcal{G}_{2,N}^{ad} = (E_{mb}T_n B_{2,N}^{ad})_{m,n \in \mathbb{Z}}$  for  $\mathcal{G}_N = (E_{mb}T_n B_N)_{m,n \in \mathbb{Z}}$  such that  $T_{\mathcal{G}_N} U_{\mathcal{G}_{1,N}^{ad}} = \mathcal{A} = T_{\mathcal{G}_N} U_{\mathcal{G}_{2,N}^{ad}}$ , where  $B_{1,N}^d$  and  $B_{2,N}^d$  are given by equations (11) and (12).

(b) Let  $\mathcal{G} = (E_{mb}T_n g)_{m,n \in \mathbb{Z}}$  be a frame for  $L^2(\mathbb{R})$  and let  $\mathcal{G}^d = (E_{mb}T_n g^d)_{m,n \in \mathbb{Z}}$  be an arbitrary dual Gabor frame of  $\mathcal{G}$ . Suppose that  $N$  is an arbitrary element of  $\mathbb{N}$  and

$$g_N^{ad} = \frac{1}{\mathbf{M}_N} S_N S_{\mathcal{G}}^{-1} g - g + S_{\mathcal{G}}(g^d).$$

Then the family  $\mathcal{G}_N^{ad} = (E_{mb}T_n g_N^{ad})_{m,n \in \mathbb{Z}}$  is an approximately dual frame of  $\mathcal{G}$  for which  $T_{\mathcal{G}} U_{\mathcal{G}_N^{ad}} = \frac{1}{\mathbf{M}_N} S_N$ .

## References

- [1] R. BALAN, Equivalence relations and distances between Hilbert frames, *Proc. Amer. Math. Soc.* **127** (1999), 2353–2366.
- [2] P. BALAZS, H. G. FEICHTINGER, M. HAMPEJS AND G. KRACHER, Double preconditioning for Gabor frames, *IEEE Trans. Signal Process.* **54** (2006), 4597–4610.
- [3] H.-Q. BUI AND R. S. LAUGESSEN, Frequency-scale frames and the solution of the Mexican hat problem, *Constr. Approx.* **33** (2011), 163–189.
- [4] O. CHRISTENSEN, Pairs of dual Gabor frames with compact support and desired frequency localization, *Appl. Comput. Harmon. Anal.* **20** (2006), 403–410.
- [5] O. CHRISTENSEN, Frames and Bases. An Introductory Course, Birkhäuser, Basel, 2007.
- [6] O. CHRISTENSEN AND R. Y. KIM, On dual Gabor frame pairs generated by polynomials, *J. Fourier Anal. Appl.* **16** (2010), 11–16.
- [7] O. CHRISTENSEN AND R. S. LAUGESSEN, Approximately dual frame pairs in Hilbert spaces and applications to Gabor frames, *Sampl. Theor. Signal Image Process.* **9** (2010), 77–89.
- [8] M. A. DEGHAN AND M. A. HASANKHANI FARD, G-dual frames in Hilbert spaces, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **75** (2013), 129–140.
- [9] R. G. DOUGLAS, On majorization, factorization and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17** (1966), 413–415.
- [10] M. DÖRFLER AND E. MATUSIAK, Nonstationary Gabor frames—approximately dual frames and reconstruction errors, *Adv. Comput. Math.* **41** (2015), 293–316.
- [11] H. G. FEICHTINGER AND K. GRÖCHENIG, Banach spaces related to integrable group representations and their atomic decomposition I, *J. Funct. Anal.* **86** (1989), 307–340.

- [12] H. G. FEICHTINGER, A. GRYBOS AND D. M. ONCHIS, Approximate dual Gabor atoms via the adjoint lattice method, *Adv. Comput. Math.* **40** (2014), 651–665.
- [13] H. G. FEICHTINGER AND N. KAIBLINGER. Varying the time-frequency lattice of Gabor frames, *Trans. Amer. Math. Soc.* **356** (2004), 2001–2023.
- [14] H. G. FEICHTINGER, D. M. ONCHIS AND C. WIESMEYR, Construction of approximate dual wavelet frames, *Adv. Comput. Math.* **40** (2014), 273–282.
- [15] J. E. GILBERT, Y. S. HAN, J. A. HOGAN, J. D. LAKEY, D. WEILAND AND G. WEISS, Smooth molecular decompositions of functions and singular integral operators, *Mem. Amer. Math. Soc.* **156** (2002), 74 pp.
- [16] M. A. HASANKHANI FARD AND M. A. DEHGHAN, Duality condition for Gabor frames  $(\chi_{[0,c]}, a, b)$  and  $(\chi_{[0,d]}, a, b)$ , *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **76** (2014), 95–98.
- [17] M. HOLSCHNEIDER, Wavelets. An analysis tool, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
- [18] A. J. E. M. JANSSEN, The duality condition for Weyl–Heisenberg frames, in: H. G. Feichtinger, T. Strohmer(Eds.), *Gabor Analysis: Theory and Applications*, Birkhäuser, Boston, 1998, 33–84.
- [19] A. J. E. M. JANSSEN, Zak transforms with few zeros and the tie, In: H. G. Feichtinger and T. Strohmer(Eds.), *Advances in Gabor Analysis*, Boston-MA. Birkhäuser 2003.
- [20] R. S. LAUGESSEN, Gabor dual spline windows, *Appl. Comput. Harmon. Anal.* **27** (2009), 180–194.
- [21] SH. LI, Y. LIU AND T. MI, Sparse dual frames and dual Gabor functions of minimal time and frequency supports, *J. Fourier Anal. Appl.* **19** (2013), 48–76.
- [22] S. LI AND D. YAN, Frame fundamental sensor modeling and stability of one-sided frame perturbation, *Acta Applicandae Mathematicae.* **107** (2009), 91–103.
- [23] G. J. MURPHY, *C\*-algebras and operator theory*, Academic Press, London 1990.
- [24] T. WERTHER, Y. C. ELDAR AND N. K. SUBBANNA, Dual Gabor frames: theory and computational aspects, *IEEE Trans. Signal Process.* **53** (2005), 4147–4158.

HOSSEIN JAVANSHIRI

Department of Mathematics, Yazd University, P.O. Box: 89195-741, Yazd, Iran

E-mail: h.javanshiri@yazd.ac.ir